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Semigroup and monoid structures of β-language

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Abstract. β-languages of order n have been introduced by the authors of [5] and their equivalence with the semi-deterministic pushdown automaton (SDPDA) languages of order n has been established in [5, 6]. In this paper, we make a study of various closure properties of β-languages. We show that the class of β-languages of order n forms a semigroup under union ($n \geq 2$) and concatenation ($n \geq 1$). We further show that the class of β-languages of order n together with the empty language $\{\lambda\}$ forms a monoid under union ($n \geq 2$) and concatenation ($n \geq 1$).

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1. Introduction

The authors of [5] introduced the concept of β-grammar and β-languages of order n and proved their equivalence with the semi-deterministic pushdown automata (SDPDA) languages of order n in [5, 6]. The class of β-languages of order n lies between non-deterministic context-free languages and deterministic context-free languages. Since the class of deterministic CFLs contains all regular languages, therefore, the class of β-languages of order n also contains all regular languages.

Again, the class of β-languages (or SDPDA languages) of order n includes the syntax of most programming languages including the mechanics

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of the parser in a typical compiler. Since each use of a production rule introduces exactly one terminal, including the null symbol "λ" into a sentential form, therefore, a string of length "k" has a derivation of at most "$(n+1)k$" steps using β-grammar of order n.

The present paper is motivated to study various closure properties of β-languages. In other words, in this paper, we study certain operations on β-languages that are guaranteed to produce again a β-language of the same order. We show that the class of β-languages of order n forms a semigroup under union ($n \geq 2$) and concatenation ($n \geq 1$). We further show that the class of β-languages of order n together with the empty languages $\{\lambda\}$ forms a monoid under union ($n \geq 2$) and concatenation ($n \geq 1$).

2. Preliminaries

In this section, we present some definitions available in the literature:

Definition 2.1 [12].

(i) A finite nonempty set Σ is called an **"alphabet"**.

(ii) A **"string"** is a finite sequence of symbols from the alphabet.

(iii) The **"concatenation"** of two strings "u" and "v" is the string obtained by appending the symbols of "v" to the right end of "u".

(iv) The **"length"** of string w denoted by $|w|$ is the number of symbols in the string.

(v) An **"empty string"** is a string with no symbol in it. It is denoted by λ and $|\lambda| = 0$.

(vi) If Σ is any alphabet, then "Σ^k"$(k \geq 0)$ denotes the set of all strings of length k with symbols from Σ.

(vii) The set of all strings over an alphabet Σ is denoted by Σ^*, i.e.

$$\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \cdots .$$

(viii) The set of all non-empty strings from the alphabet Σ is denoted by Σ^+ and is given by

$$\Sigma^+ = \Sigma^* - \{\lambda\} = \Sigma^1 \cup \Sigma^2 \cup \Sigma^3 \cup \cdots$$

(ix) A **"language"** L over an alphabet Σ is defined as a subset of Σ^*.

(x) A string in a language L is called a **"sentence"** of L.

(xi) The **"union"**, **"intersection"** and **"difference"** of two languages are defined in the set theoretic way.

(xii) The **"complement"** of a language L over an alphabet Σ is defined as $\bar{L} = \Sigma^* - L$.

(xiii) The **"concatenation"** of two languages L_1 and L_2 is the set of all strings obtained by concatenating a string of L_1 with a string of L_2, i.e.

$$L_1L_2 = \{uv|u \in L_1 \text{ and } v \in L_2\}.$$

(xiv) The **"star-closure"** of a language L is defined as

$$L^* = L^0 \cup L^1 \cup L^2 \cup \cdots .$$

Also, the **"positive-closure"** of a language L is given by

$$L^+ = L^1 \cup L^2 \cup \cdots .$$

(xv) A **"grammar"** G is defined as a quadruple

$$G = (V, T, S, P),$$

where V is a finite set of objects called **"variables"**, T is a finite set of objects called **"terminal symbols"** with $V \cap T = \phi$, $S \in V$ is a special symbol called the **"start"** symbol, P is a finite set of **"productions"** of the form $x \to y$ where $x \in (V \cup T)^+$ and $y \in (V \cup T)^*$.

(xvi) We say that the string $w = uxv$ **"derives"** the string $z = uyv$ if the string z is obtained from w by applying the production $x \to y$ to w. This is written as $w \Rightarrow z$. If

$$w_1 \Rightarrow w_2 \Rightarrow \cdots \Rightarrow w_n,$$

then we say that w_1 derives w_n and write $w_1 \Rightarrow^* w_n$.

(xvii) Let $G = (V, T, S, P)$ be a grammar. Then the **"language"** $L(G)$ generated by G is given by

$$L(G) = \{w \in T^* | S \Rightarrow^* w\}.$$

(xviii) If $w \in L(G)$, then the sequence

$$S \Rightarrow w_1 \Rightarrow w_2 \Rightarrow \cdots \Rightarrow w_n \Rightarrow w.$$

is a **"derivation"** of the sentence w. The strings $S, w_1, w_2, \cdots, w_n$ which contain variables as well as terminals are called **"sentential forms"** of the derivation.

(xix) A grammar $G = (V, T, S, P)$ is said to be **"right-linear" (resp. left-linear)** if all productions in G are of the form

$$A \to xB \text{ (resp.} A \to Bx),$$

or

$$A \to x,$$

where $A, B \in V$ and $x \in T^*$. A **"regular grammar"** is one that is either right linear or left linear.

Definition 2.2 [6]. A context-free grammar $G = (V, T, S, P)$ is said to be a **"β-grammar of order n"** ($n \geq 1$) if all productions in P are of the form $A \to ax$ where $a \in T \cup \{\lambda\}$ and $x \in V^*$ and any pair (A, a) occurs atmost "n'' times in P. A β-grammar of order n is denoted by $\beta(n)$.

Definition 2.3 [6]. The language generated by a β-grammar of order n is called a **"β-language of order n"**.

3. Closedness of β-languages of order n under union, concatenation and star-closure operations

In this section, we prove that the class of β-languages of order n is closed under union, concatenation and star-closure operations.

Theorem 3.1. *The family of β-languages of order* $n(n \geq 2)$ *is closed under union.*

Proof. Let L_1 and L_2 be two β-languages of order n ($n \geq 2$) generated by the β-grammars $G_1 = (V_1, T_1, S_1, P_1)$ and $G_2 = (V_2, T_2, S_2, P_2)$ resp. Without any loss of generality, we may assume that $V_1 \cap V_2 = \phi$ and $T_1 \cap T_2 = \phi$.

We construct a new grammar $G = (V, T, S, P)$ where

(i) $V = V_1 \cup V_2 \cup \{S\}$; S is a new variable that does not belong to V_1 and V_2,

(ii) $T = T_1 \cup T_2$, and

(iii) $P = P_1 \cup P_2 \cup \{S \to S_1;\ S \to S_2\}$.

Then G is a β-grammar of order n and $L(G)$ is a β-language of order n. It is clear that

$$L(G) = L(G_1) \cup L(G_2) = L_1 \cup L_2.$$

Thus the family of β-languages of order n ($n \geq 2$) is closed under union. □

Remark 3.2. Since the order of β-grammar G is at least 2, therefore, the result of Theorem 3.1 holds true only for $n \geq 2$.

Theorem 3.3. *The family of β-languages of order n ($n \geq 1$) is closed under concatenation.*

Proof. Let L_1 and L_2 be two β-languages of order n ($n \geq 1$) generated by the β-languages $G_1 = (V_1, T_1, S_1, P_1)$ and $G_2 = (V_2, T_2, S_2, P_2)$ resp. Without any loss of generality, we may assume that $V_1 \cap V_2 = \phi$ and $T_1 \cap T_2 = \phi$.

We construct a new grammar $G = (V, T, S, P)$ where

(i) $V = V_1 \cup V_2 \cup \{S\}$; S is a new variable that does not belong to V_1 and V_2,

(ii) $T = T_1 \cup T_2$,

(iii) $P = P_1 \cup P_2 \cup \{S \to S_1 S_2\}$.

Then G is a β-grammar of order n and $L(G)$ is a β-language of order n. Also,

$$L(G) = L(G_1)L(G_2) = L_1 L_2.$$ □

Theorem 3.4. *The class of β-languages of order n ($n \geq 2$) is closed under star-closure operation.*

Proof. Let L_1 be a β-language of order n ($n \geq 2$) generated by the β-grammar $G_1 = (V_1, T_1, S_1, P_1)$.

We construct a new grammar $G = (V, T, S, P)$ where

(i) $V = V_1 \cup \{S\}$; S is a new variable that does not belong to V,

(ii) $T = T_1$, and

(iii) $P = P_1 \cup \{S \to S_1;\ S \to \lambda\}$.

Then G is a β-grammar of order n and $L(G)$ is a β-language of order n. Also,

$$L(G) = (L(G_1))^* = L_1^*.$$

Thus the class of β-languages of order n ($n \geq 2$) is closed under star-closure operation. □

Remark 3.5. Since the order of β-grammar G is at least 2, therefore, the result of Theorem 3.4 holds true only for $n \geq 2$.

4. Semigroup and monoid structures of β-languages of order n

In this section, we discuss the semigroup and monoid structures of β-languages under union and concatenation operations. We begin with the following definition:

Definition 4.1 [4].

(i) A **"semigroup"** is a nonempty set G together with a binary operation "$*$" on G which is associative i.e.

$$a * (b * c) = (a * b) * c \text{ for all } a, b, c \in G.$$

(ii) A **"monoid"** is a semigroup G which contains a (two-sided) identity element $e \in G$ such that

$$a * e = e * a = a \text{ for all } a \in G.$$

Theorem 4.2. *The class of β-languages of order n $(n \geq 2)$ forms a semigroup under union.*

Proof. The union operation is a binary operation on the class of β-languages of order n $(n \geq 2)$. It is clearly associative since $L_1 \cup (L_2 \cup L_3) = (L_1 \cup L_2) \cup L_3$ for all β-languages L_1, L_2, L_3 of order $n(n \geq 2)$.

Thus the class of β-languages of order n $(n \geq 2)$ forms a semigroup under union. □

Theorem 4.3. *The family of β-languages of order n $(n \geq 1)$ forms a semigroup under concatenation.*

Proof. The binary concatenation operation on the class of β-languages of order n $(n \geq 1)$ is clearly associative since $L_1(L_2L_3) = (L_1L_2)L_3$ for all β-languages L_1, L_2, L_3 of order $n(n \geq 1)$.

Thus the class of β-languages of order n $(n \geq 1)$ forms a semigroup under concatenation. □

Theorem 4.4. *The class of β-languages of order n $(n \geq 2)$ together with the empty language $\{\lambda\}$ forms a monoid under union.*

Proof. Since $L \cup \{\lambda\} = \{\lambda\} \cup L = L$ for all β-languages of order $n(n \geq 2)$, therefore, the result holds in view of Theorem 4.2. □

Theorem 4.5. *The class of β-languages of order n $(n \geq 1)$ together with empty language $\{\lambda\}$ forms a monoid under concatenation.*

Proof. Since $L\{\lambda\} = \{\lambda\}L = L$ for all β-languages of order $n(n \geq 1)$, therefore, the result holds in view of Theorem 4.3. □

5. Conclusion

In this paper, we made a study of closure properties of β-languages under various operations viz. union, concatenation and star-closure. We have shown that the class of β-languages of order n forms a semigroup under union ($n \geq 2$) and concatenation ($n \geq 1$). We have further shown that the class of β-languages of order n together with the empty language $\{\lambda\}$ forms a monoid under union ($n \geq 2$) and concatenation ($n \geq 1$).

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